Grassmann-Cayley Algebra in Coq:

Part III

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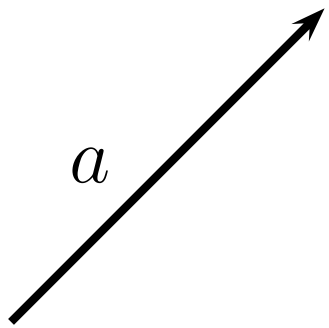
Outline

What is Grassmann-Cayley Algebra?

Application to Geometry of Incidence

Grassmann-Cayley Algebra

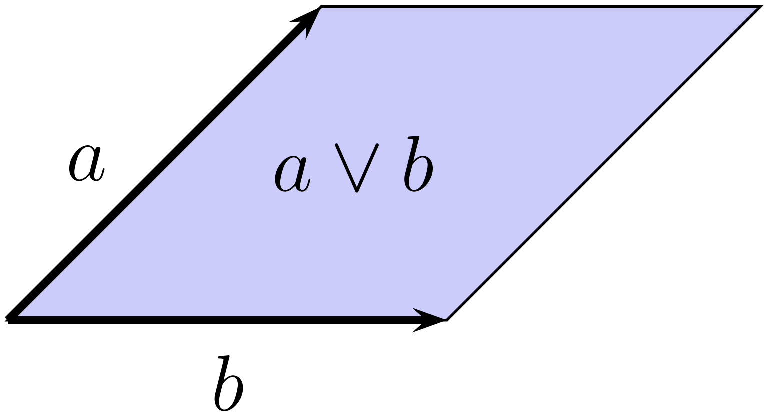


vectors: $a, b, c, \ldots$

scalar: λa

sum: $a + b$

Join Product



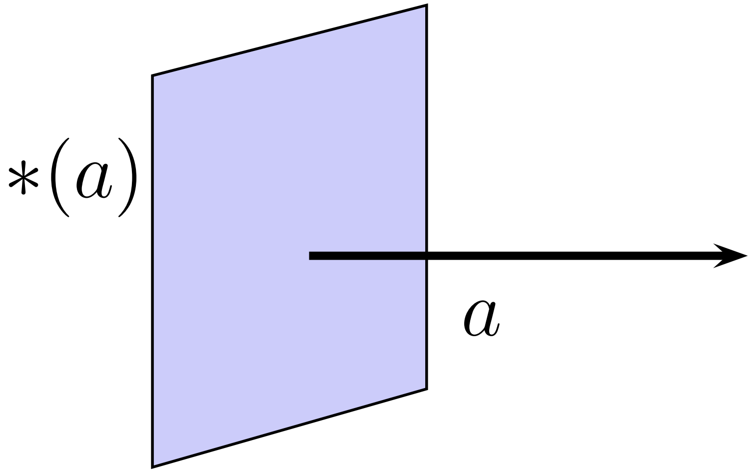
bi-vector: $a \vee b$

$$a \vee a = 0 \qquad a \vee b = -b \vee a$$

$$\lambda a \vee b = \lambda(a \vee b) \qquad (a + b) \vee c = (a \vee c) + (b \vee c)$$

Ex: $(\lambda a + \beta b) \vee (a \vee b) = 0$

Duality

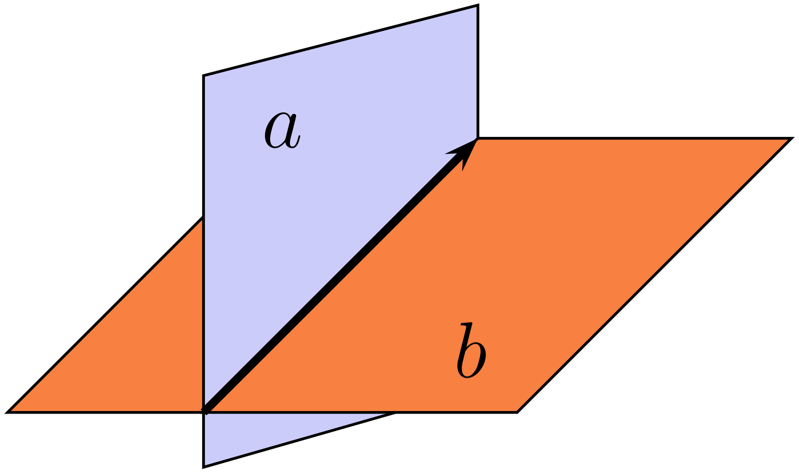


duality: $*(a)$

$*(a+b) = *(a) + *(b)$ $\qquad$ $*(\lambda(a)) = \lambda * (a)$

$a \vee *(a) = \pm E$ $\qquad$ $*(*(a)) = (-1)^{k(n+1)} a$

Meet



meet: $a \wedge b$

$$*(a \wedge b) = *(a) \vee *(b) \qquad *(a \vee b) = *(a) \wedge *(b)$$

G_3

Dimension 0: 1

Dimension 1: e_1, e_2, e_3

Dimension 2: $e_1 \vee e_2$, $e_1 \vee e_3$, $e_2 \vee e_3$

Dimension 3: $E = e_1 \vee e_2 \vee e_3$

$\mathcal{G}_3$

$\vee$	1	e_1	e_2	e_3	$e_1 \vee e_2$	$e_1 \vee e_3$	$e_2 \vee e_3$	E
1	1	e_1	e_2	e_3	$e_1 \vee e_2$	$e_1 \vee e_3$	$e_2 \vee e_3$	E
e_1	e_1	0	$e_1 \vee e_2$	$e_1 \vee e_3$	0	0	E	0
e_2	e_2	$-e_1 \vee e_2$	0	$e_2 \vee e_3$	0	$-E$	0	0
e_3	e_3	$-e_1 \vee e_3$	$-e_2 \vee e_3$	0	E	0	0	0
$e_1 \vee e_2$	$e_1 \vee e_2$	0	0	E	0	0	0	0
$e_1 \vee e_3$	$e_1 \vee e_3$	0	$-E$	0	0	0	0	0
$e_2 \vee e_3$	$e_2 \vee e_3$	E	0	0	0	0	0	0
E	E	0	0	0	0	0	0	0

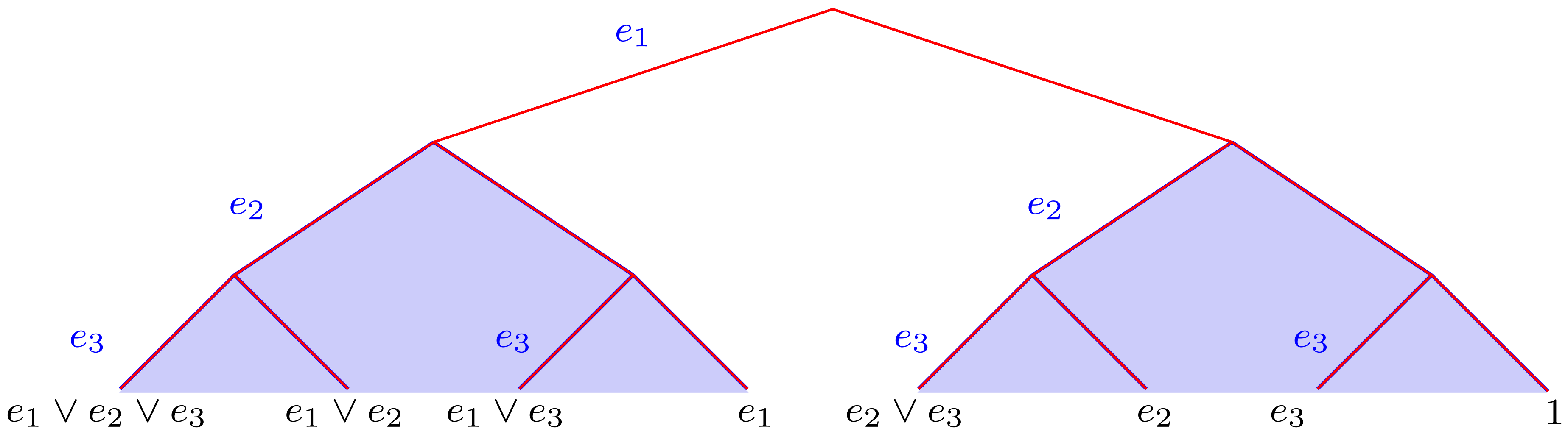
$\mathcal{G}_3$

$$
\begin{aligned}
*(1) &= E \\
*(e_1) &= e_2 \vee e_3 \\
*(e_2) &= -e_1 \vee e_3 \\
*(e_3) &= e_1 \vee e_2 \\
*(e_1 \vee e_2) &= e_3 \\
*(e_1 \vee e_3) &= -e_2 \\
*(e_2 \vee e_3) &= e_1 \\
*(E) &= 1
\end{aligned}
$$

G_3

$\wedge$	1	e_1	e_2	e_3	$e_1 \vee e_2$	$e_1 \vee e_3$	$e_2 \vee e_3$	E
1	0	0	0	0	0	0	0	1
e_1	0	0	0	0	0	0	1	e_1
e_2	0	0	0	0	0	-1	0	e_2
e_3	0	0	0	0	1	0	0	e_3
$e_1 \vee e_2$	0	0	0	1	0	e_1	e_2	$e_1 \vee e_2$
$e_1 \vee e_3$	0	0	-1	0	$-e_1$	0	e_3	$e_1 \vee e_3$
$e_2 \vee e_3$	0	1	0	0	$-e_2$	$-e_3$	0	$e_2 \vee e_3$
E	1	e_1	e_2	e_3	$e_1 \vee e_2$	$e_1 \vee e_3$	$e_2 \vee e_3$	E

Model



$$(((x_1, x_2), (x_3, x_4)), ((x_5, x_6), (x_7, x_8)))$$

Model

$$a = (((a_1, a_2), (a_3, a_4)), ((a_5, a_6), (a_7, a_8)))$$

$$b = (((b_1, b_2), (b_3, b_4)), ((b_5, b_6), (b_7, b_8)))$$

$$a \vee b =$$

$$(((a_1 b_8 + a_2 b_7 - a_3 b_6 + a_4 b_5 + a_5 b_4 - a_6 b_3 + a_7 b_2 + a_8 b_1, a_2 b_8 + a_4 b_6 - a_6 b_4 + a_8 b_2),$$
$$(a_3 b_8 + a_4 b_7 - a_7 b_4 + a_8 b_3, a_4 b_8 + a_8 b_4)),$$
$$((a_5 b_8 + a_6 b_7 - a_7 b_6 + a_8 b_5, a_6 b_8 + a_8 b_6), (a_7 b_8 + a_8 b_7, a_8 b_8)))$$

$$*(a) = (((a_8, a_7), (-a_6, a_5)), ((a_4, -a_3), (a_2, a_1)))$$

$$a \wedge b =$$

$$(((a_1 b_1, a_1 b_2 + a_2 b_1), (a_1 b_3 + a_3 b_1, a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1)),$$
$$((a_1 b_5 + a_5 b_1, a_1 b_6 + a_2 b_5 - a_5 b_2 + a_6 b_1),$$
$$(a_1 b_7 + a_3 b_5 - a_5 b_3 + a_7 b_1, a_1 b_8 + a_2 b_7 - a_3 b_6 + a_4 b_5 + a_5 b_4 - a_6 b_3 + a_7 b_2 + a_8 b_1)))$$

Model in Coq

Algebra on an arbitrary field K defined recursively on the dimension n.

Standard properties for $\vee$,$*(\;)$, $\wedge$ are all proved formally.

This gives a computational model for the algebra

Geometry of Incidence

p point:

p (element of dimension 1)

line through $[p_1, p_2]$:

$p_1 \vee p_2$

p_1 is a free point on line $[p_2, p_3]$:

$p_1 \vee (p_2 \vee p_3) = 0 \qquad (p_2 \vee p_3 \neq 0)$

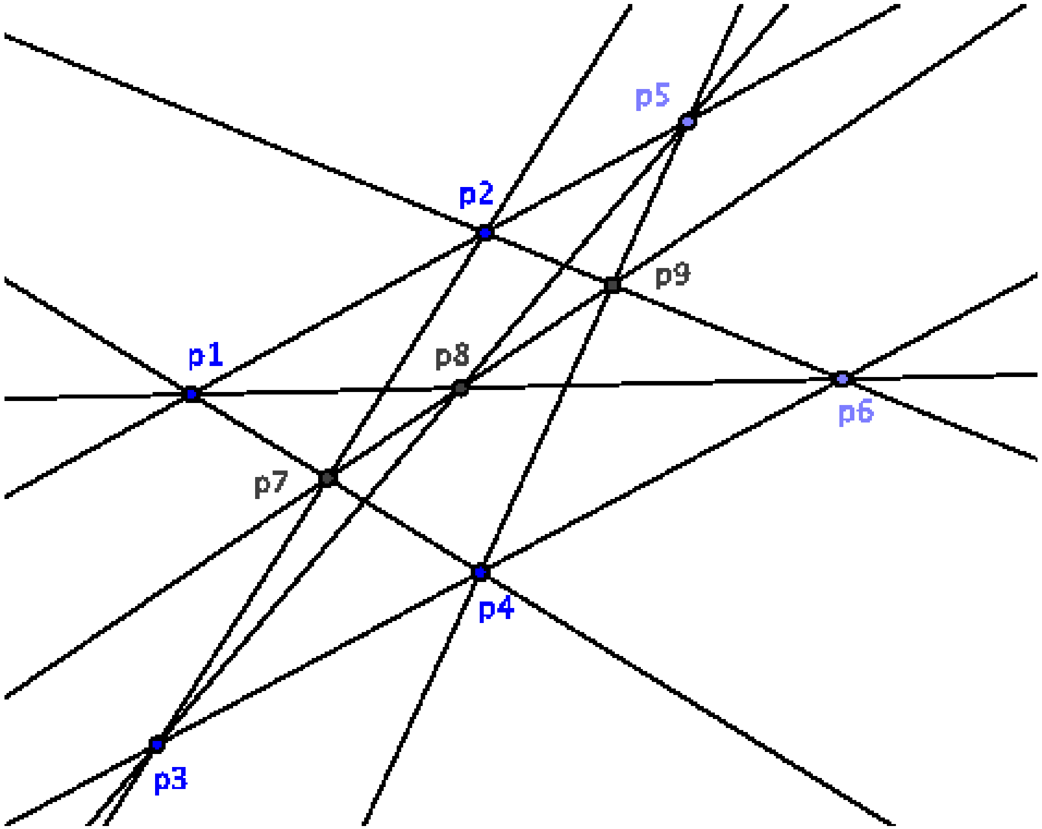
p_1 is the intersection of $[p_2, p_3]$ and $[p_4, p_5]$:

$p_1 = (p_2 \vee p_3) \wedge (p_4 \vee p_5)$

p_1 p_2 and p_3 are collinear :

$p_1 \vee p_2 \vee p_3 = 0$

Geometry of Incidence



p_5 is free on $[p_1,p_2]$

p_6 is free on $[p_3,p_4]$

p_7 is $[p_2,p_3] \cap [p_1,p_4]$

p_8 is $[p_3,p_5] \cap [p_1,p_6]$

p_9 is $[p_4,p_5] \cap [p_2,p_6]$

collinear $[p_7, p_8, p_9]$

Geometry of Incidence

$\forall\ p_1\ p_2\ p_3\ p_4\ p_5\ p_6\ p_7\ p_8\ p_9\ :\ $ `point K,`

`if` $p_1 \vee p_2 \neq 0$ `and` $p_5 \vee p_1 \vee p_2 = 0$ `and`
$p_3 \vee p_4 \neq 0$ `and` $p_6 \vee p_3 \vee p_4 = 0$ `and`
$p_7 = p_2 \vee p_3 \wedge p_1 \vee p_4$ `and`
$p_8 = p_3 \vee p_5 \wedge p_1 \vee p_6$ `and`
$p_9 = p_4 \vee p_5 \wedge p_2 \vee p_6$

`then` $p_7 \vee p_8 \vee p_9 = 0$

Geometry of Incidence

How to Prove this?

Interactively:
M. Hawrylycz,
"A geometric identity for Pappus' theorem"

Automatically:
H. Li and Y. Wu,
Automated short proof generation for projective geometric theorems with Cayley and bracket algebras: I. Incidence geometry

Bracket Algebra

Express everything in term of $[p_i, p_j, p_k]$

$[\alpha p_i, \beta p_i, p_j] = 0$ $\qquad$ $[p_i, p_j, p_k] = -[p_j, p_i, p_k]$

$p_i \vee p_j \vee p_k = [p_i, p_j, p_k]E$

collinear p_i, p_j and p_k $\iff$ $[p_i, p_j, p_k] = 0$.

Bracket Algebra

Elimination of an intersection

if p_i is the intersection of $[p_j, p_k]$ and $[p_l, p_m]$ then

$$[p_i, p_n, p_o] = -[p_j, p_n, p_o][p_k, p_l, p_m] + [p_k, p_n, p_o][p_j, p_l, p_m].$$

$\longrightarrow \qquad \mathbb{Z}[[p_i, p_j, p_k], \ldots]$

Bracket Algebra

Elimination of a point on a line

p_i is free on [p_j,p_k]

$$p_j \vee p_k \neq 0 \quad \text{and} \quad p_i \vee p_j \vee p_k = 0$$

$$\exists \alpha\, \beta, \quad p_i = \alpha p_j + \beta p_k.$$

if $\quad p_i$ free on [p_j,p_k] $\quad$ then

$$[p_i, p_l, p_m] = \alpha[p_j, p_l, p_m] + \beta[p_k, p_l, p_m].$$

$$\longrightarrow \qquad \mathbb{Z}[\alpha, \beta, \ldots][[p_i, p_j, p_k], \ldots]$$

Reduction

Plucker relations

Contradiction:

$$[p_i, p_j, p_k][p_i, p'_j, p'_k] - [p_i, p_j, p'_k][p_i, p'_j, p_k] - [p_i, p_j, p'_j][p_i, p_k, p'_k] = 0$$

Reduction

Ordered bracket: $[p_i, p_j, p_k][p'_i, p'_j, p'_k]$

when $p_i > p_j > p_k$, $p'_i > p'_j > p'_k$ and $p_i \leq p'_i$, $p_j \leq p'_j$, $p_j \leq p'_j$

splitting: $p_j > p'_j$

$$[p_i, p_j, p_k][p'_i, p'_j, p'_k] =$$
$$[p_i, p'_i, p'_j][p_j, p_k, p'_k] - [p_i, p'i, p'k][p_j, p_k, p'_j] + [p_i, p'_j, p'_k][p_j, p_k, p'_i]$$

splitting: $p_j \leq p'_j$ and $p_k > p'_k$

$$[p_i, p_j, p_k][p'_i, p'_j, p'_k] =$$
$$[p_i, p_j, p'_i][p_k, p'_j, p'_k] - [p_i, p_j, p'_j][p_k, p'_i, p'_k] + [p_i, p_j, p'_k][p_k, p'_i, p'_j]$$

Future Works

Make use of the computational model

Show that our encoding of G_3 is a model for incidence geometry

Extend the decision procedure for G_n