Formalizing a Discrete Model of the Continuum in Coq from a Discrete Geometry Perspective

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Motivations

- Constructions and Computations in Geometry
  - Exact Real Computations vs. Approximations
  - How to Handle Continuous Objects in a Discrete Setting ?
- So far, most Formal Proofs do not take Computations/Coordinates Issues into Account.
  - Proj. Geometry : Magaud, Narboux, Schreck (ADG'2008)
  - Geometric Algebras : Fuchs, Thery (ADG'2010)
- Computing is interesting,
  Reasoning about such Computations is even better
  - Correctness Issues ?

Context and Related Work

- Exact Geometric Computation (Yap, . . .)
  - A Pragmatic Approach
  - Only recently, some Considerations about the Foundations of the Calculus
- Computable Analysis (Weihrauch, . . .)
  - Foundations of Computation on Real Numbers
  - Correctness Properties of the Calculus

Our Goal

- Working on the Foundational Side...
- ...but with an Effective Model based on Integers
- Getting the Best of the Two Worlds :
  at the same time a Foundational and Practical Approach
- Applications
  - Geometric Algebras
    (Thery, Fuchs : ADG'2010)
  - Exact Real Functions Representation
    (Chollet, Wallet, Fuchs et al. : IWCIA 2009)
  - Discrete Ellipsis Connectivity
    (Chollet, Wallet, Andres et al. : CompIMAGE 2010)

The continuum onto a computer ?

Working relatively to a given scale

- At a given scale ; points are of a specified size.
- We can use as many scales as necessary.
- To obtain enough points between two points ; it is always possible to change the scale.

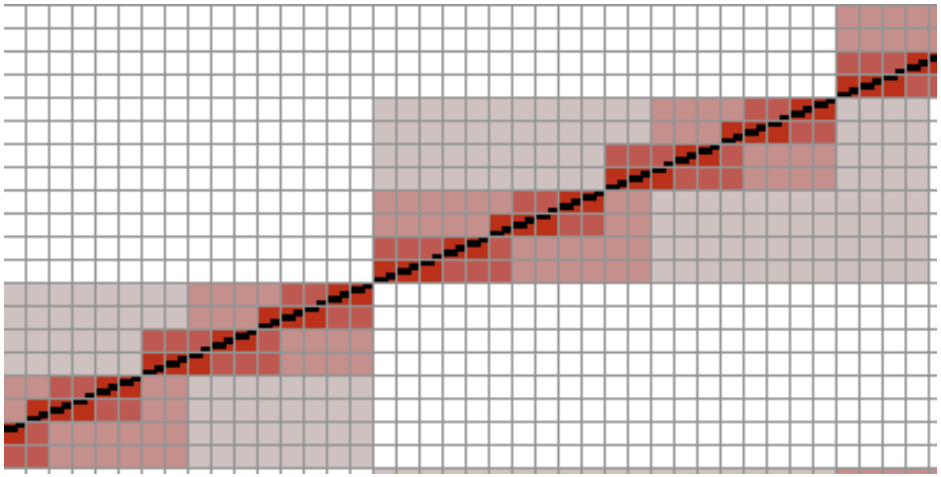


FIG.: A line of slope 1/2 represented at different scales.

The continuum onto a computer ?

What is used

- A constructive axiomatic of the real line (Bridges, 1999)
  - A way to define what are the real numbers that can be computed.
- A model of the continuum (Harthong-Reeb, 1984)
  - In order to define what could be the continuum in the discrete world.
- A nonstandard arithmetic (Laugwitz-Schmieden $\approx$ 60, Martin-Löf $\approx$ 80)
  - Define what is large and what is small.

A discrete model of the continuum

Obtain enough numbers between two numbers

- Choose an Ω-number ω infinitely large as new unity $\omega \overset{def}{\equiv} 1_\omega$.
- Hence, between two integer numbers, there is *as many* integers you want.

The Harthong-Reeb line

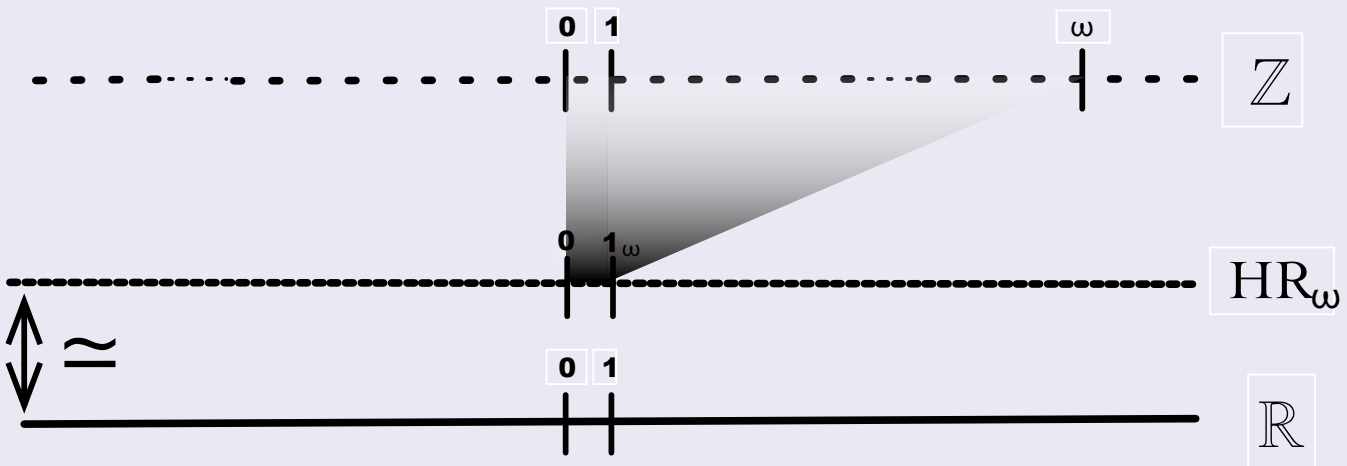
$$\mathcal{HR}_\omega = \{X \in \mathbb{Z}_\Omega,\ \exists n \in \mathbb{N},\ |X| \leq n\omega\}$$

$\mathcal{HR}_\omega$ is a rescaling over $\mathbb{Z}_\Omega$.

A discrete model of the continuum

The real line $\mathbb{R}$ is similar to the discrete line $\mathbb{Z}$ seen from far away.

The Harthong-Reeb line



A discrete model of the continuum

The Harthong-Reeb line

Let X and Y be 2 elements of $\mathcal{HR}_\omega$.

- $X =_\omega Y \Leftrightarrow$ for all n in $\mathbb{N}$, $n|X - Y| \leq \omega$
- $X >_\omega Y \Leftrightarrow$ exists an n in $\mathbb{N}$, $n(X - Y) \geq \omega$
- $X +_\omega Y =_{def} X + Y$.
- $X \times_\omega Y =_{def} \lfloor XY/\omega \rfloor$.
- $0_\omega =_{def} 0$ and $1_\omega := \omega$.
- $-_\omega X =_{def} -X$.
- If X is such that $X \neq_\omega 0 \quad X^{(-1)_\omega} =_{def} \left\lfloor \frac{\omega^2}{X} \right\rfloor$.

An axiomatic presentation of the constructive real line

- In 1999, Douglas Bridges introduced an axiomatic presentation of the constructive real line.
  - It is a system $(R, +, \times, =, >, 0, 1, \mathrm{Opp}, \mathrm{Inv})$ which satisfies a list of 17 axioms.
  - Let us call a Bridges-Heyting ordered field any system which satisfies these axioms.
- These axioms are organized into 3 groups :
  - The first is dedicated to the algebraic operations,
  - The second to the order structure
  - The third to the usual Archimedes' axiom and to a constructive least-upper bound principle.

$\mathcal{HR}_\omega$ is a Bridges-Heyting ordered field

Theorem

The Harthong-Reeb line is a Bridges-Heyting ordered field.

Here, the Harthong-Reeb line denotes the complete system

$$(\mathcal{HR}_\omega, +_\omega, \times_\omega, =_\omega, >_\omega, 0_\omega, 1_\omega, \mathrm{Opp}_\omega, \mathrm{Inv}_\omega)$$

- This result shows that the Harthong-Reeb line is a nonstandard model of the constructive real line.
- We formalize it in the Coq proof assistant.

Laugwitz-Schmieden

Definition

- Let us consider the sequences $a = (a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{Z}$.
- equipped with the following equality :

$$a = b \text{ if there exists } N \in \mathbb{N} \text{ s.t. } \forall n > N,\ a_n = b_n.$$

- An Ω-*integer* a is an equivalence class for this equality. We denote the set of Ω-integers by $\mathbb{Z}_\Omega$.

Examples

- $(2, 2, 2, 2, 2, 2, \ldots)$ denotes the Ω-integer 2.
- $(1, 5, 4, 2, 2, 2, \ldots) = (2, 2, 2, 2, 2, 2, \ldots)$ are in the same equivalence class.

Basic Operations for Laugwitz-Schmieden Integers

Operations and relations on $\mathbb{Z}_\Omega$:

- $a + b =_{def} (a_n + b_n)$ and $-a =_{def} (-a_n)$ and $a \times b =_{def} (a_n \times b_n)$;
- $a > b =_{def} [(\exists N \, \forall n > N) \; a_n > b_n]$ and $a \geqslant b =_{def} [(\exists N \, \forall n > N) \; a_n \geqslant b_n]$;
- $|a| =_{def} (|a_n|)$.

Two classes of elements :

- $a = (a_n)_{n \in \mathbb{N}}$ is standard if $\exists p \in \mathbb{Z}$ such that $\exists N \in \mathbb{N}, \; \forall n > N, \; a_n = p$ otherwise a is nonstandard.
- $a = (a_n)_{n \in \mathbb{N}}$ is infinitely large if (a_n) is increasing ($\lim a_n \simeq +\infty$).

What we get is a nonstandard theory of the arithmetic (that was defined by Laugwitz et Schmieden).

Computation Rules

Operations and relations on $\mathbb{Z}_\Omega$:

- $a + b =_{def} (a_n + b_n)$ and $-a =_{def} (-a_n)$ et
  $a \times b =_{def} (a_n \times b_n)$;
- $a > b =_{def} [(\exists N \, \forall n > N) \; a_n > b_n]$ et
  $a \geqslant b =_{def} [(\exists N \, \forall n > N) \; a_n \geqslant b_n]$;
- $|a| =_{def} (|a_n|)$.

Two classes of elements :

- $a = (a_n)_{n \in \mathbb{N}}$ is standard if $\exists p \in \mathbb{Z}$ s.t.
  $\exists N \in \mathbb{N}, \; \forall n > N, \; a_n = p$
- $a = (a_n)_{n \in \mathbb{N}}$ is infinitely large if $(\lim a_n \simeq +\infty)$

Limitations

- The usual properties true on $\mathbb{Z}$ are not always verified for Laugwitz-Schmieden Integers. For instance

$$(\forall a, b \in \mathbb{Z}_\Omega) \quad (a \geqslant b) \vee (b \geqslant a) \tag{1}$$

  is not valid : e.g. take $a = ((-1)^n)_{n\in\mathbb{N}}$ and $b = ((-1)^{n+1})_{n\in\mathbb{N}}$.
- Laugwitz-Schmieden is not an actual model of the NS Integers we use to build the $\mathcal{HR}_\omega$ line.
- Nevertheless we can prove

$$(\forall n \in \mathbb{N}) \quad (a_n \geqslant b_n) \vee (b_n \geqslant a_n) \tag{2}$$

  because we have a decidability property for $\mathbb{Z}$.

Formalization in Coq

- The Coq Proof Assistant :
  Interactive Theorem Proving and Program Development
  - Formal Definitions of Concepts
  - Formal Proofs of Properties
- Formalizing using Modules
  - A Module Specification : NS Integers
  - A Functor defining Harthong-Reeb Line on top of it
  - A (partial) Implementation of NS Integers - Laugwitz-Schmieden

A Theory of Non-Standard Integers

- Basic Constructions and Operations

```
Parameters A:Type  a0:A  a1:A.

Parameter plusA : A -> A -> A.
Parameter leA : A -> A -> Prop. (?<=)
Parameter absA : A-> A.
```

- Properties

```
Parameter plus_neutral : forall x,0 + x = x.

Lemma Ath : ring_theory a0 a1 plusA multA ...
Add Ring A_ring : Ath (abstract).

Parameter abs_triang :
  forall x y, |x+y| ?<= |x| + | y |.
```

Non Standard Features

We have an element `w`, which is positive.
The predicate `lim` is defined throught the following rules :

(LIM1) *The integer 1 is limited.*

(LIM2) *The sum and the product of two limited integers are limited.*

(LIM3) *There exists integers which are not limited.*

(LIM4) *If X is limited and $|Y| \leq |X|$, then Y is itself limited.*

Building the Harthong-Reeb Line $\mathcal{HR}_\omega$

```
Definition P := fun (x:A)=>
  exists n:A, (lim n /\ 0 ?< n /\  (|x| ?<= n*w)).

Definition HRw := {x:A | P x }.

Lemma Padd : forall x y, P x -> P y -> P ( x + y).
unfold P; intros x y Hx Hy.
elim Hx [...]
Qed.

Definition HRwplus (x y: HRw) : HRw :=
match x with exist xx Hxx =>
match y with exist yy Hyy =>
exist P (xx + yy) (Padd xx yy Hxx Hyy)
end end.
```

Algebraic Axioms (R1)

The group (R1) deals with the algebraic structure :

(R1.1) $x + y = y + x$

(R1.2) $(x + y) + z = x + (y + z)$

(R1.3) $0 + x = x$

(R1.4) $x + (-x) = 0$

(R1.5) $xy = yx$

(R1.6) $x(yz) = (xy)x$

(R1.7) $1x = x$

(R1.8) $xx^{-1} = 1$ for $x \neq 0$

(R1.9) $x(y + z) = xy + xz$

The second group of axioms (R2)

The second group (R2) deals with the properties of the order and links it to the previously-proved algebraic properties :

(R2.1) $\neg((x > y) \wedge (y > x))$

(R2.2) $(x > y) \Rightarrow \forall z((x > z) \vee (z > y))$

(R2.3) $\neg(x \neq y) \Rightarrow x = y$

(R2.4) $(x > y) \Rightarrow \forall z((x + z) \vee (y + z))$

(R2.5) $((x > 0) \wedge (y > 0)) \Rightarrow xy > 0$

The third group of axioms (R3)

Special Properties of the order relation :

(R3.1) *Archimedes Axiom.*
For all $x \in R$, there exists $n \in \mathbb{Z}$ s.t. $n > x$.

(R3.2) *The constructive Upper-bound principle.* Let S be a non-empty subset of $\mathcal{HR}_\omega$ s.t.

- $\exists b \in \mathcal{HR}_\omega \; \forall s \in S \;\; b \geq s$
- $\forall \alpha, \beta \in \mathcal{HR}_\omega \; (\beta > \alpha) \Rightarrow (\forall s \in S \;\; \beta \geq s) \vee (\exists s \in S \;\; s > \alpha)$

Then, there exists $b \in \mathcal{HR}_\omega$ which is an upper bound of S :

- $\forall s \in S \;\; b \geq s$
- $\forall b' \; (b > b') \Rightarrow (\exists s \in S \;\; s > b')$

Proof Techniques and Adequacy

- Symbolic computation requires more precise proofs.
  Let x be an integer (in $\mathbb{Z}$, $\mathbb{N}$ or $\mathtt{A}$?)
- It avoids shortcuts, e.g. `HRwdiff` / `HRwequal`
- Modules and modules types to distinguish between theories and models of such theories.
- Proof-irrelevance : Objects of HR_ω are equal regardless of the way the property P is shown
- High-level tools to avoid technical proofs : `Add Ring`
- Formalizing helps improving the writing of informal proofs.

Summary

- Non Standard : a real theory to talk about infinitesimals.
- Approximations are replaced by Infinitesimals.
- A scalable framework : one can change its point of view to have as many points as we want inbetween 2 given points of the line (this is achieved by changing the scale).
- A comprehensive framework to reason about geometric constructions
- Formalizing in Coq improves the confidence in the proofs, makes them more precise.

Future Work

- Completing the Formalization
  (esp. w.r.t. the upper bound property)
- A really constructive model : Martin-Löf model
- Applications to Discrete Geometry
  - Proving formally formulas used to compute the intersection of discrete lines are sound.
  - Discrete Circles