Formalizing Projective Geometry in Coq

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Outline

Related Work

- Gilles Kahn [Kah95]
- Christophe Dehlinger, Jean-François Dufourd and Pascal Schreck [DDS00]
- Laura Meikle and Jacques Fleuriot [MF03]
- Frédérique Guilhot [Gui05]
- Julien Narboux [Nar04, Nar07]
- Jean Duprat [Dup02]
- Bezem and Hendriks [BH08]

Motivations

Projective Geometry ?

- Only considers incidence properties.
- In 2D, two lines always intersect.
- A simple but powerful framework
- Flats, ranks and a proof of Desargues theorem
- Further work: a certified prover for collinearity/planarity properties (à la Schreck) and a certified formal solver

Plane Projective Geometry

- Objects: Points, Lines
- An Incidence relation between Points and Lines
- We assume decidability of Incidence

$$\forall P : Point, \forall l : Line, \{Incid\ P\ l\}\{\neg Incid\ P\ l\}$$

- Decidability for Point (resp. Line) equality is proved !

Axioms for Projective Plane Geometry

- Axiom Line Existence

$$\forall A\ B : Point, (\exists l : Line, A \in l \wedge B \in l)$$

- Axiom Point Existence

$$\forall l\ m : Line, (\exists A : Point, A \in l \wedge A \in m)$$

- Axiom Line Unicity

$$\forall A\ B : Point, A \neq B \Rightarrow$$
$$\forall l\ m : Line, A \in l \wedge B \in l \wedge A \in m \wedge B \in m \Rightarrow l = m$$

- Axiom Point Unicity

$$\forall l\ m : Line, l \neq m \Rightarrow$$
$$\forall A\ B : Point, A \in l \wedge A \in m \wedge B \in l \wedge B \in m \Rightarrow A = B$$

More Axioms

Axiom Uniqueness (subsumes both unicity axioms)

$$\forall A\ B : Point, \forall l\ m : Line,$$
$$A \in l \Rightarrow B \in l \Rightarrow A \in m \Rightarrow B \in m \Rightarrow A = B \vee l = m$$

Axiom Four Points

$$\exists A : Point, \exists B : Point, \exists C : Point, \exists D : Point,$$
$$distinct4\ A\ B\ C\ D \wedge$$
$$(\forall l : Line, (A \in l \wedge B \in l \Rightarrow C \notin l \wedge D \notin l) \wedge$$
$$(A \in l \wedge C \in l \Rightarrow B \notin l \wedge D \notin l) \wedge$$
$$(A \in l \wedge D \in l \Rightarrow B \notin l \wedge C \notin l) \wedge$$
$$(B \in l \wedge C \in l \Rightarrow A \notin l \wedge D \notin l) \wedge$$
$$(B \in l \wedge D \in l \Rightarrow A \notin l \wedge C \notin l) \wedge$$
$$(C \in l \wedge D \in l \Rightarrow A \in l \wedge B \in l))$$

Another system

Alternative Axioms (replacing Axiom Four Points)

- Axiom Three Points

  $$\forall l : Line, \\ \exists ABC : Point, (A \neq B \wedge B \neq C \wedge A \neq C) \wedge A \in l \wedge B \in l \wedge C \in l$$

- Axiom Lower Dimension

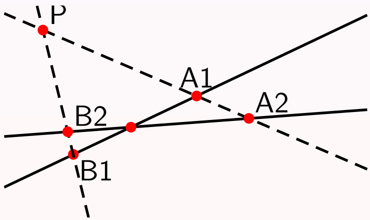
  $$\exists l_1 : Line, \exists l_2 : Line, l_1 \neq l_2$$

Properties

- Equivalence of the two axiom systems
- Decidability of point (resp. line) equality derived from decidability of incidence :

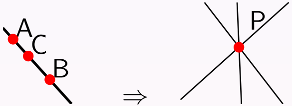
$$\forall A : Point, \forall l : Line, \{Incid\ A\ l\} + \{\neg Incid\ A\ l\}$$

- Flats and their characterization (in a couple of minutes. . .)
- Lines as sets of points and bijections

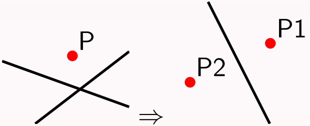


Duality

- A functor from Projective Plane Geometry to itself
- Application: proving some theorem automatically
  - 3 points on a line $\rightarrow$ 3 lines through a point

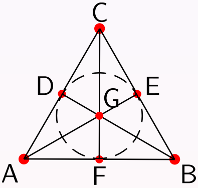


  - 1 point outside l_1 and l_2 $\rightarrow$ 1 line not through P_1 and P_2



Models

- Finite Models
  $PG(dim, b)$ where given a point on a line, b is the number of other lines through the point.
  - Fano's plane ($= PG(2,2)$ the smallest projective plane)



  - PG(2,5) (31 points and as many lines, thus 923 521 cases)
- Infinite Model: Homogeneous Coordinates
- Remark : this raises scalability issues in 3D

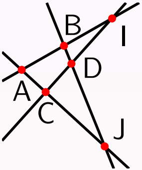
Axioms for Space Projective Geometry (I)

Axiom Line Existence

$$\forall A\ B : Point, \exists l : Line, A \in l \wedge B \in l$$

Axiom Pasch (*replaces* Axiom Point Existence)

$$\forall ABCD : Point, \forall l_{AB}\ l_{CD}\ l_{AC}\ l_{BD} : Line, dist4\ ABCD \wedge$$
$$A \in l_{AB} \wedge B \in l_{AB} \wedge C \in l_{CD} \wedge D \in l_{CD} \wedge$$
$$A \in l_{AC} \wedge C \in l_{AC} \wedge B \in l_{BD} \wedge D \in l_{BD} \wedge$$
$$(\exists I : Point, I \in l_{AB} \wedge I \in l_{CD}) \Rightarrow$$



$$\exists J : Point, (J \in l_{AC} \wedge J \in l_{BD})$$

Axiom Uniqueness

$$\forall A\ B : Point, \forall l\ m : Line,$$
$$A \in l \Rightarrow B \in l \Rightarrow A \in m \Rightarrow B \in m \Rightarrow$$
$$A = B \vee l = m$$

Axioms for Space Projective Geometry (II)

Axiom Three Points

$\forall l : Line, \exists ABC : Point, (A \neq B \wedge B \neq C \wedge A \neq C) \wedge$
$A \in l \wedge B \in l \wedge C \in l$

Axiom Lower Dimension

$\exists l_1 l_2 : Line, \neg \exists p : Point, p \in l_1 \wedge p \in l_2$

Axiom Upper Dimension

$\forall l1 l2 l3 : Line, dist3\ l1 l2 l3 \rightarrow$
$\exists l4 : Line, \exists J1 : Point, \exists J2 : Point, \exists J3 :$
$Point, (dist3\ J1\ J2\ J3) \wedge$
$(Intersect\_In l1\ l4\ J1) \wedge (Intersect\_In\ l2\ l4\ J2) \wedge$
$(Intersect\_In\ l3\ l4\ J3)$

Flats and Ranks (I)

- Sets as their characteristic function ($Point \rightarrow Prop$)
- Definition
  A flat is a set of points such that the entire line defined by A and B lies in the flat whenever A and B belong to it.
  $flat(v) = \forall AB : Point, v\ A \rightarrow v\ B \rightarrow A \neq B \rightarrow$
  $\forall l : Line, Incid\ A\ l \rightarrow Incid\ B\ l \rightarrow \forall C : Point, Incid\ C\ l \rightarrow v\ C$
- The empty set $\emptyset$, singletons, lines, planes and the whole space are flats:
  - Empty: $fun(p : Point) \Rightarrow False$
  - Singleton x: $fun(p : Point) \Rightarrow p = x$
  - Line l: $fun(p : point) \Rightarrow p \in l$
  - Plane $l_1\ l_2\ l_1 \neq l_2\ l_1 \bigcap l_2 \neq \emptyset$: $fun(x : Point) \Rightarrow$
    $\exists l : Line, x \in l \wedge \exists I, \exists J, I \neq J \wedge$
    $Intersect\_In\ l\ l1\ I \wedge Intersect\_In\ l\ l2\ J$
  - Space: $fun(p : Point) \Rightarrow True$

Flats and Ranks (II)

- Characterization of flats
  The only flats in dim. 3 are the empty set, singletons, lines, planes and the whole space.
- Rank of a set of points is defined by flat-closure of this set. This flat is either the empty set (rk=0), or a singleton set (rk=1), or a line (rk=2), etc.
- Informally, rank corresponds to dim $+ 1$. It ranges from 0 ($\emptyset$) to 4 (the whole space).
- Collinearity or Planarity expressed with ranks : examples
  - rk(A,B)=1 means A equals B
  - rk(A,B,C)=2 means A, B and C are collinear with at least 2 distinct
  - rk(A,B,C,D)=3 means A, B, C and D are coplanar with at least 3 distinct
  - rk(A,B,C,D)=4 means A,B, C and D are not coplanar and all distinct

Matroid Properties for Rank

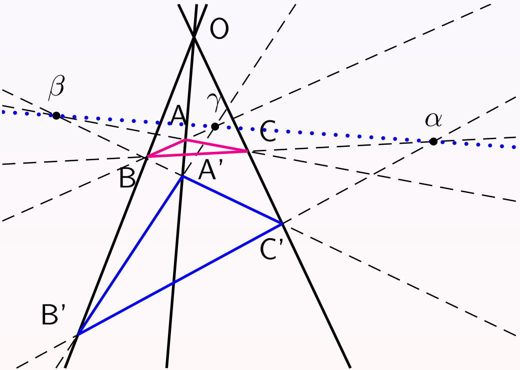
- Rank verifies matroid properties.
- Properties

$$
\begin{aligned}
&(1)\ rk(\emptyset) = 0 \\
&(2)\ e \subset e' \rightarrow rk(e) \leq rk(e') \\
&(3)\ rk(e' \bigcup e) + rk(e' \bigcap e) \leq rk(e') + rk(e)
\end{aligned}
$$

- scale ? we can proceed with one singleton at a time (rather than using (3)
- What about $rk(x) = 1$, $rk(x, y) = 2$ and $rk(x, y, z) = 3$?

Proving Desargues' with ranks (I)

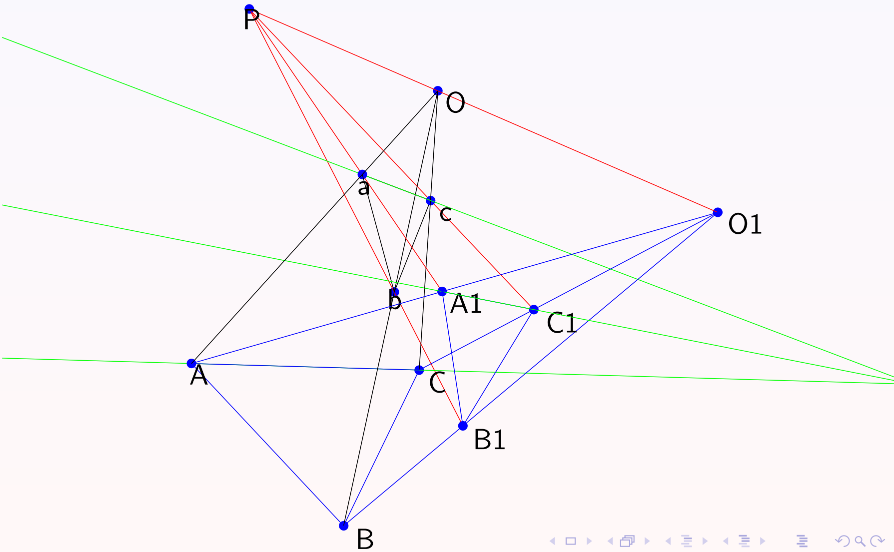
A 3D version of Desargues' theorem



Proving Desargues' with ranks (II)

- Formal Statement
  $rk(a, A, O) = 2, rk(b, B, O) = 2, rk(c, C, O) = 2$ (perspective)
  $rk(a, b, c) = 3,\ rk(A, B, C) = 3$ (2 triangles)
  $rk(A, B, C, a, b, c) = 4$ (not coplanar)
  $rk(A, B, \gamma) = 2, rk(a, b, \gamma) = 2$ (building γ)
  $rk(A, \beta, C) = 2, rk(a, \beta, c) = 2$ (building β)
  $rk(\alpha, B, C) = 2, rk(\alpha, b, c) = 2$ (building α)
  $rk(M, N) = 2\ \forall MN \in \{a, b, c, A, B, C\}\ where\ M \neq N$
  $\rightarrow rk(\alpha, \beta, \gamma) = 2$
- Lifting into 3D one of the triangles
- Projecting back is not required as α, β and γ are on the 3 relevant lines.

Desargues' theorem



Main Results

- Equivalence of axiom systems
- Decidability of equality (both for points and lines)
- Duality
- Models in 2D (Fano, PG(2,5))
- Flats characterization in 2D and 3D
- Matroid structure for rank
- Desargues' theorem

Next steps

- Write more proofs (currently around 10000 lines)
- Write more tactics to solve goals automatically: successful experiments have been carried out and need improvement
- Complete Desargues' proof (3D lifting, particular cases)
- In Plane PG, prove Pappus implies Desargues (Hessenberg's theorem)

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