Formalizing projective geometry in Coq

Nicolas Magaud Julien Narboux Pascal Schreck

Université de Strasbourg, France

GDR LTP, 21 Octobre 2009, Université Paris XII

Outline

1. Objectives
2. Formal proofs in geometry
3. Projective geometry
4. A case study : Desargues' theorem
5. Formalization in Coq
6. Future work and conclusion

Geometric contraint solving and automated deduction in geometry

1. Proofs of theorems in geometry are needed for solving geometric contraint systems.
2. Most methods in automated deduction in geometry (Wu's method, the area method) can only deal with geometric statements expressed as a ruler and compass *construction*.

We need a system to prove both mathematical theorems and programs!

Related work

Geometry

- Hilbert's Grundlagen [DDS00, MF03]
- Tarski's geometry [Nar07]
- High-school geometry [Gui05]
- Hessenberg's theorem [BH08]
- Convex-hulls algorithm [MF05] [PB01]
- Image segmentation algorithm [Duf07]

Degenerated cases

- *Using Three-Valued Logic to Specify and Verify*, Brandt and Schneider [BS05]

Using matroids

- Incidence constraints, a combinatorial approach, Schreck-Michelucci[MS06]

Why projective geometry ?

- Non degeneracy conditions
- A simple framework

Axiom system (2D)

Line-Existence

$$\forall A\ B : Point, (\exists l : Line, A \in l \wedge B \in l)$$

Point-Existence

$$\forall l\ m : Line, (\exists A : Point, A \in l \wedge A \in m)$$

Axiom Uniqueness

$$\begin{array}{l} \forall A\ B : Point, \forall l\ m : Line, \\ A \in l \Rightarrow B \in l \Rightarrow A \in m \Rightarrow B \in m \Rightarrow A = B \vee l = m \end{array}$$

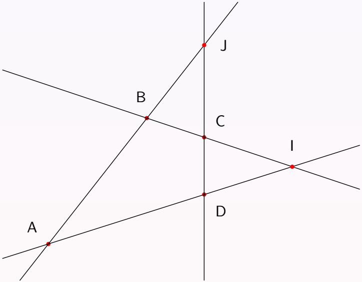
Axiom Four Points

$$\begin{array}{l} \exists A : Point, \exists B : Point, \exists C : Point, \exists D : Point, \\ distinct4\ A\ B\ C\ D\ \wedge \\ (\forall l : Line, (A \in l \wedge B \in l \Rightarrow C \notin l \wedge D \notin l) \wedge \\ (A \in l \wedge C \in l \Rightarrow B \notin l \wedge D \notin l) \wedge \\ (A \in l \wedge D \in l \Rightarrow B \notin l \wedge C \notin l) \wedge \\ (B \in l \wedge C \in l \Rightarrow A \notin l \wedge D \notin l) \wedge \\ (B \in l \wedge D \in l \Rightarrow A \notin l \wedge C \notin l) \wedge \\ (C \in l \wedge D \in l \Rightarrow A \in l \wedge B \in l)) \end{array}$$

Axiom system ($\geq 3D$)

Line-Existence $\forall A\ B : Point, \exists l : Line, A \in l \wedge B \in l$

Pasch
$\forall A\ B\ C\ D : Point, \forall l_{AB}\ l_{CD}\ l_{AC}\ l_{BD} : Line,$
$A \neq B \wedge A \neq C \wedge A \neq D \wedge B \neq C \wedge B \neq D \wedge C \neq D \wedge$
$A \in l_{AB} \wedge B \in l_{AB} \wedge C \in l_{CD} \wedge D \in l_{CD} \wedge$
$A \in l_{AC} \wedge C \in l_{AC} \wedge B \in l_{BD} \wedge D \in l_{BD} \wedge$
$(\exists I : Point, I \in l_{AB} \wedge I \in l_{CD}) \Rightarrow$
$(\exists J : Point, J \in l_{AC} \wedge J \in l_{BD})$



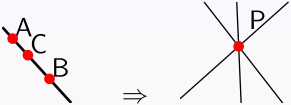
Uniqueness $\forall A\ B : Point, \forall l\ m : Line,$
$A \in l \wedge B \in l \wedge A \in m \wedge B \in m \Rightarrow A = B \vee l = m$

Three-Points $\forall l : Line, \exists A\ B\ C : Point,$
$A \neq B \wedge B \neq C \wedge A \neq C \wedge A \in l \wedge B \in l \wedge C \in l$

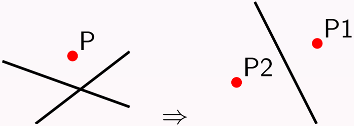
Lower-Dimension $\exists l\ m : Line, \forall p : Point, p \notin l \vee p \notin m$

Duality

- Proving some theorems by duality:
    - 3 points on a line ⇒ 3 lines through a point



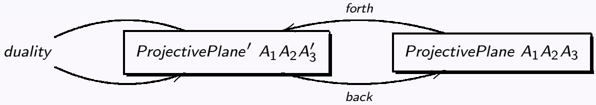
    - 1 point outside l_1 and l_2 ⇒ 1 line not through P_1 and P_2



- The solution: a functor from Projective Plane Geometry to itself

Implementation in Coq !

- Module types (boxes) and Functors (arrows)



- Duality as a functor

```
Module swap (M': ProjectivePlane') <: ProjectivePlane'.
Definition Point := M'.Line.
Definition Line := M'.Point.
Definition Incid := fun (x:Point) (y:Line) => M'.Incid y x.
[...]
Definition a1_exist := M'.a2_exist.
Definition a2_exist := M'.a1_exist.
[...]
```

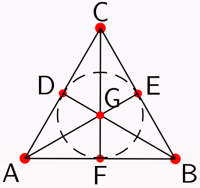
- Application

```
Lemma outsider : forall l1 l2: Line, { P:Point | ~Incid P l1/\~Incid P l2}.
Proof. [...] Qed.

Lemma dual_example :
forall P1 P2 : Point,{l : Line | ~ Incid P1 l /\ ~ Incid P2 l}.
Proof. apply ProjectivePlaneFacts_m.outsider. Qed.
```

Models

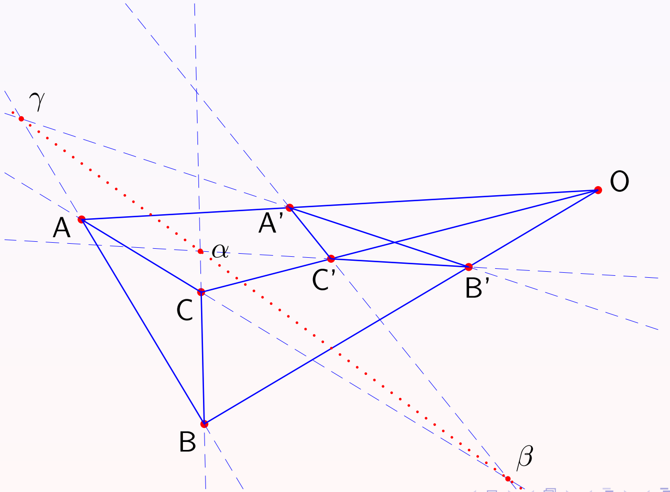
- Finite Models
  $PG(dim, b)$ where given a point on a line, b is the number of other lines through the point.
  - Fano's plane ($= PG(2,2)$ the smallest projective plane)



  - PG(2,5) (31 points and as many lines, thus 923 521 cases)
- Infinite Model: Homogeneous Coordinates

Desargues' theorem

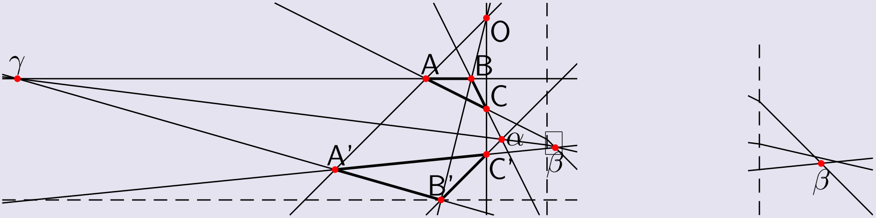
let E be a 3D or higher projective space, if the 3 lines joining the corresponding vertices of triangles ABC and $A'B'C'$ all meet in a point O, then the 3 intersections of pairs of corresponding sides α, β and γ lie on a line.



Why 3D ?

Desargue's theorem is false in some 2D models.

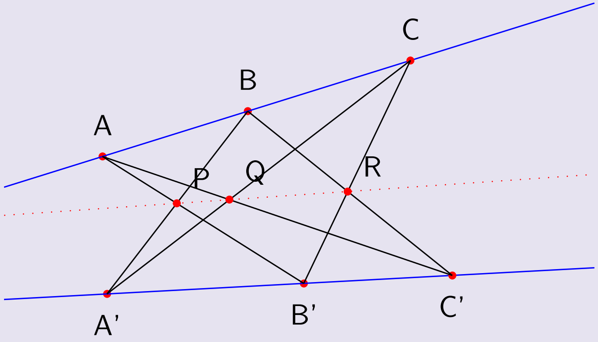
Moulton's plane



Hessenberg's theorem

If Pappus holds then Desargues holds as well.

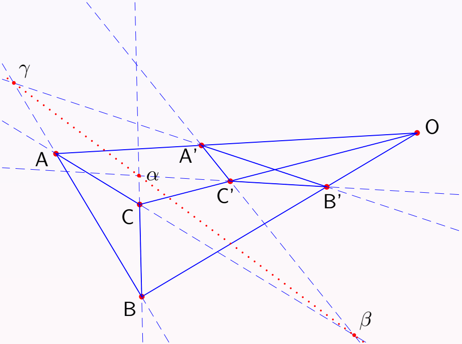
Pappus



The proof

Outline of the proof:

- We prove the 3D version of Desargues' theorem.
- We lift the 2D statement to a 3D version of Desargues' theorem which projects into the 2D statement.



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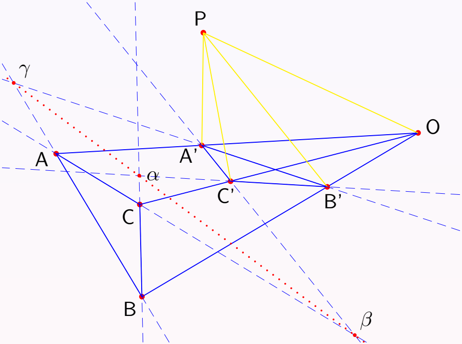

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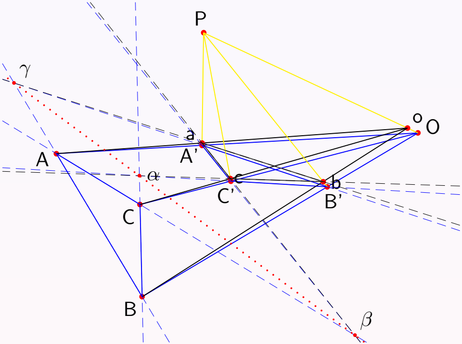

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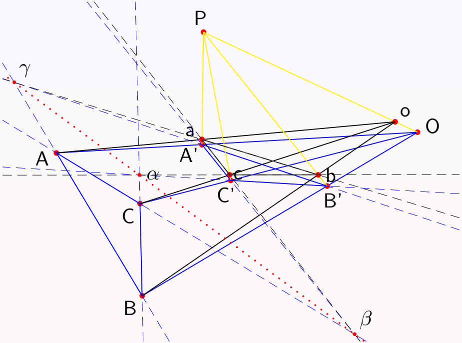

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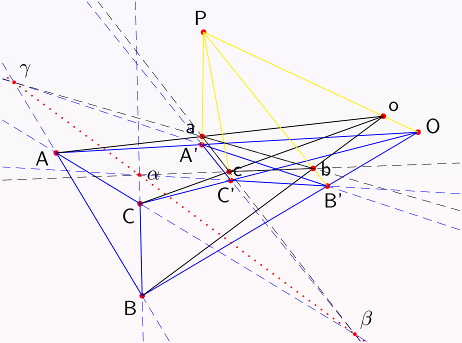

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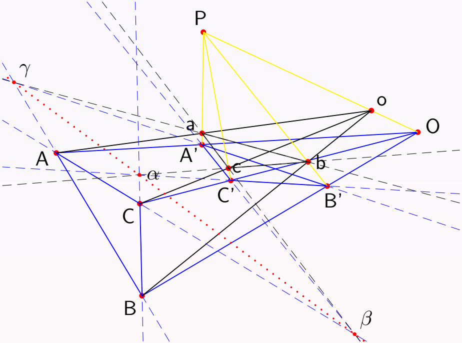

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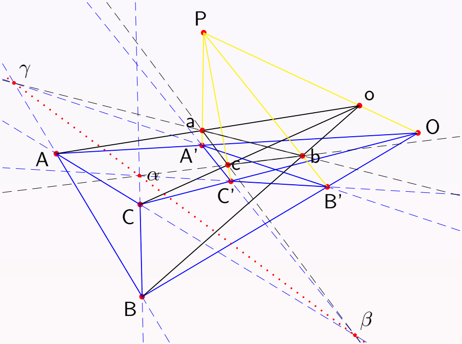

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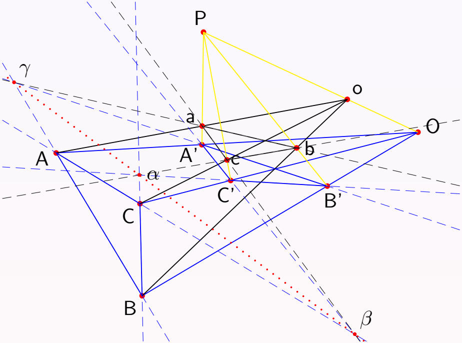

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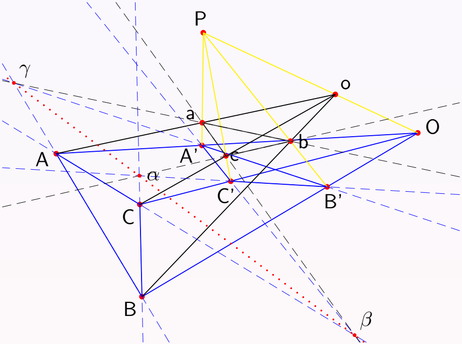

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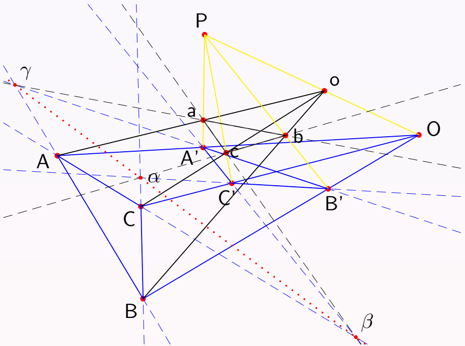

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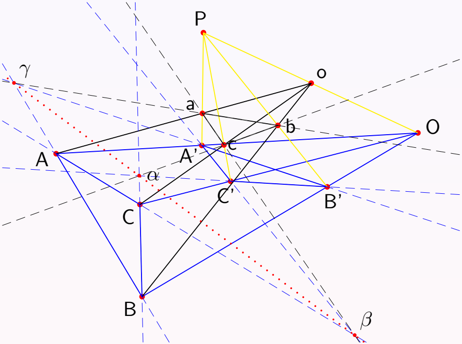

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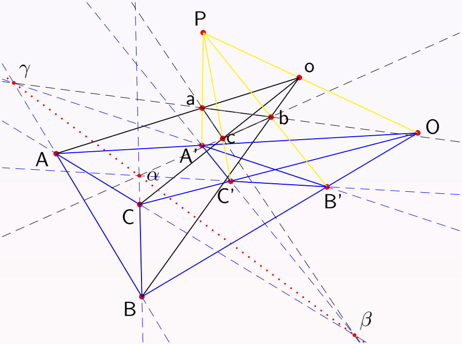

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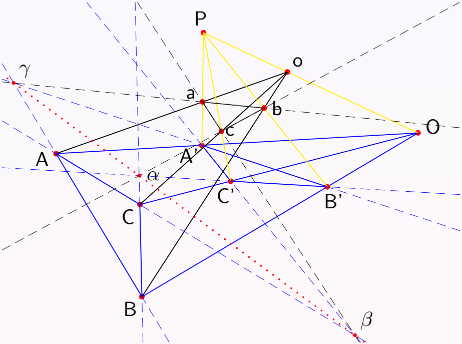

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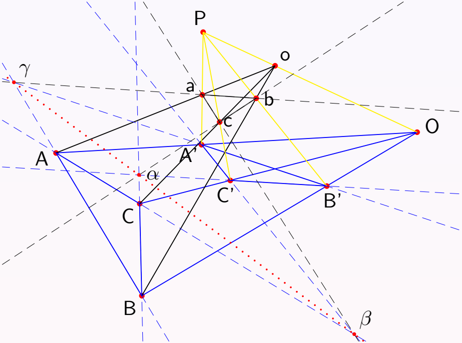

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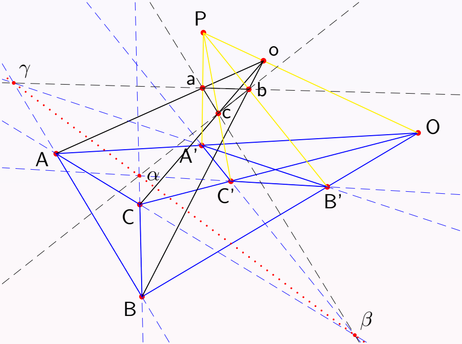

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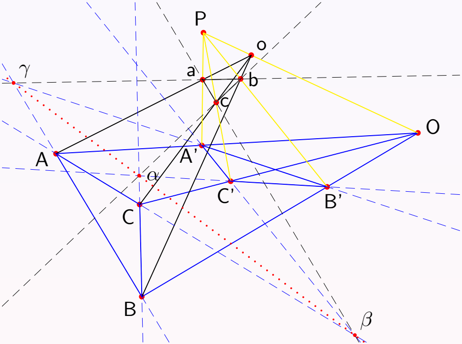

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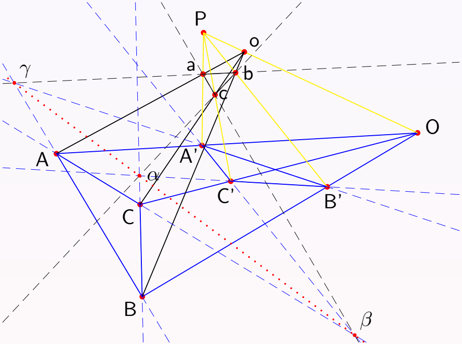


Formalization in Coq

We use the concept of rank:

R1 $\forall X \subseteq E, 0 \leq rk(X) \leq |X|$ (nonnegative and subcardinal)

R2 $\forall XY \subseteq E, X \subseteq Y \Rightarrow rk(X) \leq rk(Y)$ (nondecreasing)

R3 $\forall XY \subseteq E, rk(X \cup Y) + rk(X \cap Y) \leq rk(X) + rk(Y)$ (submodular)

Example

$rk\{A, B\} = 1$	$A = B$
$rk\{A, B\} = 2$	$A \neq B$
$rk\{A, B, C\} = 2$	A, B, C are collinear with at least two of them distinct
$rk\{A, B, C\} \leq 2$	A, B, C are collinear
$rk\{A, B, C\} = 3$	A, B, C are not collinear
$rk\{A, B, C, D\} = 3$	A, B, C, D are co-planar, not all collinear
$rk\{A, B, C, D\} = 4$	A, B, C, D are not co-planar
$rk\{A, B, C, D, E\} \leq 2$	A, B, C, D, E are all collinear

Axiom system to capture projective at least 3-dimensional space

Rk-Singleton $\forall P : Point, rk\{P\} \geq 1$

Rk-Couple $\forall P\ Q : Point, P \neq Q \Rightarrow rk\{P, Q\} \geq 2$

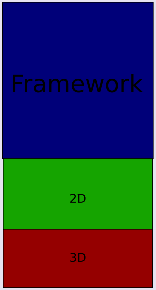
Rk-Pasch $\forall A\ B\ C\ D, rk\{A, B, C, D\} \leq 3 \Rightarrow \exists J, rk\{A, B, J\} = rk\{C, D, J\} = 2$

Rk-Three-Points $\forall A\ B, \exists C, rk\{A, B, C\} = rk\{B, C\} = rk\{A, C\} = 2$

Rk-Lower-Dimension $\exists A\ B\ C\ D, rk\{A, B, C, D\} \geq 4$

Outline of the proof

1. general lemmas (framework)
2. proof of the 3D statement
3. proof of the 2D statement by projection



Proof technique

Using axiom R3

$$\forall XY, rk(X \cup Y) + rk(X \cap Y) \leq rk(X) + rk(Y)$$

$$\Downarrow$$

$$rk\{A, B, C, D, I\} + rk\{\quad I\} \quad \leq \quad rk\{A, B, I\} + rk\{C, D, I\}$$

Proof technique

Using axiom R3

$$\forall XY, rk(X \cup Y) + rk(X \cap Y) \leq rk(X) + rk(Y)$$

$$\not\Downarrow$$

$$rk\{A, B, C, D, I\} + rk\{A, I\} \quad \leq \quad rk\{A, B, I\} + rk\{C, D, I\}$$

This is false. If for instance $A = C$, $\{A, B, I\} \cap \{C, D, I\} = \{A, I\}$

Proof technique

Using axiom R3

$$\forall XY, rk(X \cup Y) + rk(X \cap Y) \leq rk(X) + rk(Y)$$

$$\not\Downarrow$$

$$rk\{A, B, C, D, I\} + rk\{A, I\} \quad \leq \quad rk\{A, B, I\} + rk\{C, D, I\}$$

This is false. If for instance $A = C$, $\{A, B, I\} \cap \{C, D, I\} = \{A, I\}$

Lemma (R3-ALT)

$$\forall XYI, I \subseteq X \cap Y \Rightarrow rk(X \cup Y) + rk(I) \leq rk(X) + rk(Y)$$

Conclusion

- A formalization of duality
- Models
- An axiom system to capture projective geometry based on ranks.
- A way to express incidence relations thanks to ranks.
- A case study: Desargues.
- $\geq$ 15k lines of Coq

Future work

- Automatic proofs using hexamys.

Marc Bezem and Dimitri Hendriks.
On the Mechanization of the Proof of Hessenberg's Theorem in Coherent Logic.
J. of Automated Reasoning, 40(1):61–85, 2008.

Jens Brandt and Klaus Schneider.
Using three-valued logic to specify and verify algorithms of computational geometry.
In *ICFEM*, volume 3785 of *LNCS*, pages 405–420. Springer-Verlag, 2005.

Christophe Dehlinger, Jean-François Dufourd, and Pascal Schreck.
Higher-Order Intuitionistic Formalization and Proofs in Hilbert's Elementary Geometry.
In *ADG'00*, volume 2061 of *LNAI*, pages 306–324. Springer-Verlag, 2000.

Jean-François Dufourd.
Design and Formal Proof of a New Optimal Image Segmentation Program with Hypermaps.
Pattern Recognition, 40(11):2974–2993, 2007.

Frédérique Guilhot.
Formalisation en Coq et visualisation d'un cours de géométrie pour le lycée.
TSI, 24:1113–1138, 2005.
In french.

Laura Meikle and Jacques Fleuriot.
Formalizing Hilbert's Grundlagen in Isabelle/Isar.
In *Theorem Proving in Higher Order Logics*, pages 319–334, 2003.

Laura Meikle and Jacques Fleuriot.
Mechanical theorem proving in computation geometry.
In Hoon Hong and Dongming Wang, editors, *ADG'04*, volume 3763 of *LNAI*, pages 1–18. Springer-Verlag, 2005.

Dominique Michelucci and Pascal Schreck.
Incidence Constraints: a Combinatorial Approach.
International J. of Computational Geometry and Application, 16(5-6):443–460, 2006.

Julien Narboux.
Mechanical theorem proving in Tarski's geometry.
In *ADG'06*, volume 4869 of *LNAI*, pages 139–156. Springer-Verlag, 2007.

David Pichardie and Yves Bertot.
Formalizing convex hulls algorithms.
In *TPHOLs'2001*, volume 2152 of *LNCS*, pages 346–361. Springer-Verlag, 2001.