La droite réelle de Harthong-Reeb, un modèle d'une droite réelle constructive ?

A. Chollet[2] G. Wallet[2] L. Fuchs[1]
G. Largeteau-Skapin[1] E. Andres[1]

[1]XLIM-SIC
CNRS UMR 6172
Université de Poitiers

[2]LMA
Université de La Rochelle

Strasbourg - Galapagos 2008
17-18 Décembre 2008

Introduction

Reveilles' discrete line

- Everybody knows Reveillès' discrete analytical line

$$0 \leq ax - by < \omega$$

- That led to the Discrete Analytical Geometry

Back to the origins

- But the origins of the theoretical framework are mostly forgotten
  - They are based on a nonstandard theory of analysis (IST, E. Nelson).
  - G. Reeb and J. Harthong; nonstandard model of continuum based on integers.
  - Compute solutions of differential equations using infinitesimal concept.

Introduction

This work

- Revisit Reeb's team work under the light of discrete geometry.
- Insist on the algorithmic nature of Reeb's approach (constructive mathematics).

Fallouts

- Provide an access to the work of Reeb's team
- Improve the method that has led to the Reveillès' discrete analytical line.
- Explore and improve the theoretical framework.
- Hope that it will help to define new discrete differential concepts.

Introduction

Introduce an infinitely large number

"Les entiers naïfs ne remplissent pas $\mathbb{N}$"[a] *(G. Reeb)*

- Naive integers; $0, 1, 2, \ldots$, if $a \in \mathbb{N}$ is naive so is $a + 1$.
- There exists non naive integers; $a = 0$ if every even integer > 2 is a sum of two primes or a is the first even integer that is not a sum of two primes. Is a naive?
- If $\omega \in \mathbb{N}$ is not naive then $\omega > 1, \ldots, \omega > 10^{10^{10^{10}}}, \ldots$
- $\omega \in \mathbb{N}$ is infinitely large, it is not a standard integer. First stone of the Harthong-Reeb model.

[a] "Naive integers don't fill $\mathbb{N}$"

Introduction

Nonstandard approach leads to algorithms

"L'étonnante richesse des vues intuitionnistes [...] ne pouvait qu'aboutir [...] à une version de l'analyse nonstandard"[a] *(from G. Reeb)*

- Constructive (intuitionist) mathematics are computational mathematics
- In computer science constructive approach leads to the correspondance

Maths		Logic		Programming
proof	=	term	=	program
theorem	=	type	=	specification

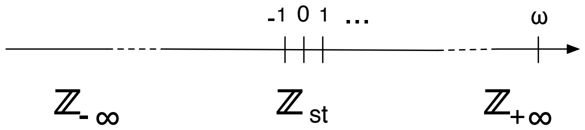
[a] "The surprising richness of intuitionistic approaches could only end in a version of nonstandard analysis"

Outline

1. The Harthong-Reeb line as a numerical system
   - Presentation of the Harthong-Reeb line
   - The birth of the Reveillès discrete analytical line
2. The constructive content of the Harthong-Reeb line
   - Some points on constructivism
   - Constructive aspects of the Harthong-Reeb line
3. Les Ω-entiers de Laugwitz
   - Critique de la constructivité de $\mathcal{HR}_\omega$
   - La théorie de Laugwitz
   - Critique de la constructivité de $\mathcal{HR}_\Omega$

We need some infinitely large number ω

- Starting from Reeb's slogan "*Naive integers don't fill* $\mathbb{N}$", we are led to imagine some number $\omega \in \mathbb{N}$ which is greater than each naive number $0, 1, 2, 3, \ldots$



- A surprising point is that this "nonstandard dream" is consistent with usual formal theory of $\mathbb{N}$.
- Consequently, we can enrich formal arithmetic by the introduction of a weak axiomatic form of nonstandard analysis which is well suited for our purpose.

A minimal form of nonstandard analysis

We introduce a new predicate *st* over integer numbers: $st(x)$ "means" that the integer x is standard. This predicate is external to the classical integer theory and its meaning directly derives from the following axioms ANS1, ANS2, ANS3, ANS4:

- ANS1. *The number 1 is standard.*
- ANS2. *The sum and the product of two standard numbers are standard.*
- ANS3. *There are nonstandard integer numbers.*
- ANS4. *For all $(x, y) \in \mathbb{Z}^2$ such that x is standard and $|y| \leq |x|$, the number y is standard.*

From now on, we choose an integer ω which is nonstandard. It is clear that ω is greater than each standard number: we say that $\omega \simeq +\infty$.

The Harthong-Reeb formal system

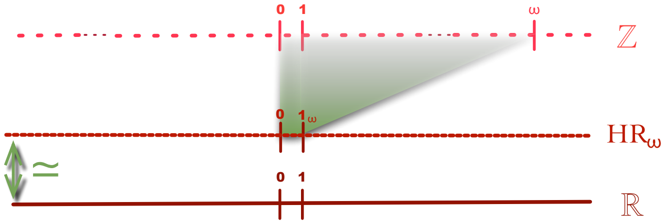
Definition

The Harthong-Reeb line is the set $\mathcal{HR}_\omega$ of $X \in \mathbb{Z}$ such that $\exists^{st} n \in \mathbb{N} \quad |X| \leq n\omega$, with the relations and operations:

- $X =_\omega Y \quad \Leftrightarrow \quad \forall^{st} n \in \mathbb{N} \quad n|X - Y| \leq \omega$.
- $Y >_\omega X \quad \Leftrightarrow \quad \exists^{st} n \in \mathbb{N} \quad n(Y - X) \geq \omega$.
- $X +_\omega Y := X + Y$.
- $X \times_\omega Y := \left\lfloor \frac{X.Y}{\omega} \right\rfloor$.
- $0_\omega := 0$ and $1_\omega := \omega$.
- $\forall Z \in \mathcal{HR}_\omega \quad \mathrm{Opp}_\omega(Z) := -Z$ and $\mathrm{Inv}_\omega(Z) := \left\lfloor \frac{\omega^2}{Z} \right\rfloor$.

The meaning of $\mathcal{HR}_\omega$

$\mathcal{HR}_\omega$ is a vision of $\mathbb{Z}$ at the scale ω. That means that ω is interpreted as the unit of the new system. We also have $0 =_\omega 1 =_\omega \ldots =_\omega k =_\omega \lfloor\sqrt{\omega}\rfloor$ for all standard $k \in \mathbb{Z}$.



Thus $\mathcal{HR}_\omega$ is very like the real line $\mathbb{R}$. Each real number x (like π or $\sqrt{2}$) is represented in $\mathcal{HR}_\omega$ by the integer $\lfloor x\,\omega\rfloor$.

Théorèmes

Théorème de continuité

La droite d'Harthong-Reeb est isomorphe à l'ensemble des réels muni d'une égalité d'infini proximité.

◁ On utilise les fonctions suivantes:

$$\begin{array}{rccl} \varphi_\omega : & \mathcal{HR}_\omega & \longrightarrow & \mathbb{R}_{lim} \\ & X & \longmapsto & \varphi_\omega(X) := \dfrac{X}{\omega} \end{array}$$

$$\begin{array}{rccl} \psi_\omega : & \mathbb{R}_{lim} & \longrightarrow & \mathcal{HR}_\omega \\ & x & \longmapsto & \psi_\omega(x) := \lfloor x\omega \rfloor \end{array}$$

▷

Arithmetization of Euler scheme in $\mathcal{HR}_\omega$

Let $\left\{ x(a) = b,\ \dfrac{dx}{dt} = f(t,x) \right\}$ be a differential equation.

The corresponding Euler scheme in $\mathbb{R}$ with step $1/\beta$ is:

$$\left\{ t_0 = a,\ x_0 = b,\ t_{n+1} = t_n + \frac{1}{\beta},\ x_{n+1} = x_n + \frac{1}{\beta} f(t_n, x_n) \right\}$$

Choosing an integer $\beta \leq \omega$ such that $\beta \simeq +\infty$, we get in $\mathcal{HR}_\omega$, an arithmetized scheme at the scale ω:

$$(\star) \begin{cases} T_0 = \lfloor a\omega \rfloor,\ X_0 = \lfloor \omega b \rfloor \\ T_{n+1} = T_n + \omega \div \beta \\ X_{n+1} = X_n + F(T_n, X_n) \div \beta \end{cases}$$

where $F(T_n, X_n) = \left\lfloor \omega f\left(\frac{T_n}{\omega}, \frac{X_n}{\omega} \right) \right\rfloor$ (arithmetization of $f(t,x)$)

Interpretation in an intermediate scale

Following an idea of Reeb, it is interesting to interpret the solution (T_n, X_n) of $(\star)$ in an intermediate scale. For that, we choose $\omega = \beta^2$ and we interpret (T_n, X_n) in $\mathcal{HR}_\beta$.

Every $Z \in \mathcal{HR}_\omega$ is written $Z = \overline{Z}\beta + \widehat{Z}$ where $\overline{Z} \in \mathcal{HR}_\beta$ and $\widehat{Z} \in \{0, \ldots, \beta - 1\}$. With these new variables, we get the arithmetization of the Euler scheme calculated at the scale β^2 and interpreted at the scale β:

$$(\star\star) \begin{cases} \overline{T}_0 = \lfloor a\beta^2 \rfloor \div \beta, \ \overline{X}_0 = \lfloor b\beta^2 \rfloor \div \beta \text{ and } \widehat{X}_0 = \lfloor b\beta^2 \rfloor \bmod \beta \\ \overline{T}_{n+1} = \overline{T}_n + 1 \\ \overline{X}_{n+1} = \overline{X}_n + (\widehat{X}_n + \overline{F}_n) \div \beta \\ \widehat{X}_{n+1} = (\widehat{X}_n + \overline{F}_n) \bmod \beta \end{cases}$$

where $\overline{F}_n = F(\overline{T}_n\beta + \lfloor a\beta^2 \rfloor \bmod \beta, \overline{X}_n\beta + \widehat{X}_n) \div \beta$.

Towards the Reveillès discrete line

We apply the preceding arithmetization process to a continuous line $x = ct + d$ interpreted as the solution of the trivial equation $\{x(0) = d, x' = c\}$.

$$(\overline{T}_n, \overline{X}_n) = \left(n, \left\lfloor \frac{\lfloor \omega d \rfloor + n(C \div \beta)}{\beta} \right\rfloor \right)$$

It is the graph of a discrete function of the following form

$$X(T) = \left\lfloor \frac{L + TK}{\beta} \right\rfloor$$

equivalent to the condition

$$0 \leq KT - \beta X + L < \beta$$

in the spirit of the Reveillès discrete analytical line.

Constructive mathematics

Nonstandard approach leads to algorithms

"L'étonnante richesse des vues intuitionnistes [...] ne pouvait qu'aboutir [...] à une version de l'analyse nonstandard"[a] *(from G. Reeb)*

[a] "The surprising richness of intuitionistic approaches could only end in a version of nonstandard analysis"

- Constructive mathematics have their origin into the criticisms of formal mathematics (Brouwer 1907)
  - We must take care of the use of logical rules with infinite sets.
- Heyting's work ($\approx$ 1930) shows that constructive mathematics are based on intuitionistic logic (no $A \vee \neg A$)
  - The meaning of a formula is the set of its proofs.

Constructive mathematics

Constructive mathematics and programming

The difference between constructive mathematics and programming [...] lies in that programs must be written in a formal notation [...] whereas, in constructive mathematics [...] the computational procedures are normally left implicit in the proofs [...]. (P. Martin-Löf)

- In 1967 Bishop's book develops large parts of analysis:
  - Real numbers are unspecified algorithms that produce arbitrary rational approximation.
  - Equality between real numbers is defined.
- The Curry-Howard correspondence ($\approx$ 1980):

$$\begin{array}{ccccc} \text{proof} & = & \text{term} & = & \text{program} \\ \text{theorem} & = & \text{type} & = & \text{specification} \end{array}$$

  gives the way to link proofs and programs.

An axiomatic presentation of the constructive real line

- In 1999, Douglas Bridges introduced an axiomatic presentation of the constructive real line.
  - It is a system $(R, +, \times, =, >, 0, 1, \mathrm{Opp}, \mathrm{Inv})$ which satisfies a list of 17 axioms.
  - Let us call a Bridges-Heyting ordered field any system which satisfies these axioms.
- These axioms are organized into 3 groups:
  - The first is dedicated to the algebraic operations,
  - The second to the order structure
  - The third to the usual Archimedes' axiom and to a constructive least-upper bound principle.

$\mathcal{HR}_\omega$ is a Bridges-Heyting ordered field

The main result of our work is the following theorem.

Theorem

The Harthong-Reeb line is a Bridges-Heyting ordered field.

Here, the Harthong-Reeb line denotes the complete system

$$(\mathcal{HR}_\omega, +_\omega, \times_\omega, =_\omega, >_\omega, 0_\omega, 1_\omega, \mathrm{Opp}_\omega, \mathrm{Inv}_\omega)$$

This result shows that the Harthong-Reeb line is a nonstandard model of the constructive real line.

Two important points about the logical framework of our proof:

- Our treatment of the relations $=_\omega$ and $>_\omega$ complies with the constructive rules (i.e intuitionistic logic).
- We identify the standard integers (the elements of $\mathbb{Z}_{\mathrm{st}}$) with the usual (constructive) integers.

Example of proof: $(X \times_\omega Y) \times_\omega Z =_\omega X \times_\omega (Y \times_\omega Z)$

From the definition, $(X \times_\omega Y) \times_\omega Z = \left\lfloor \left\lfloor \frac{X.Y}{\omega} \right\rfloor \frac{Z}{\omega} \right\rfloor$.

$$\left\lfloor \frac{XYZ}{\omega^2} \right\rfloor + \left\{ \frac{XYZ}{\omega^2} \right\} - \left\{ \frac{XY}{\omega} \right\} \frac{Z}{\omega} - \left\{ \left\lfloor \frac{X.Y}{\omega} \right\rfloor \frac{Z}{\omega} \right\}$$

Since $Z \in \mathcal{HR}_\omega$, $\exists^{st} n \in \mathbb{N}$ such that $|Z| \leq n\omega$. Hence

$$\left| \left\{ \frac{XYZ}{\omega^2} \right\} - \left\{ \frac{XY}{\omega} \right\} \frac{Z}{\omega} - \left\{ \left\lfloor \frac{X.Y}{\omega} \right\rfloor \frac{Z}{\omega} \right\} \right| \leq n + 2$$

and thus, $(X \times_\omega Y) \times_\omega Z =_\omega \left\lfloor \frac{XYZ}{\omega^2} \right\rfloor$.

A similar treatment would give $X \times_\omega (Y \times_\omega Z) =_\omega \left\lfloor \frac{XYZ}{\omega^2} \right\rfloor$.

The third group of axioms on $\mathcal{HR}_\omega$

Axiom of Archimedes.

It is a direct consequence of the definition of $X \in \mathcal{HR}_\omega$.

Constructive least-upper bound principle. Let $S \subset \mathcal{HR}_\omega$ such that:

- $\exists b \in \mathcal{HR}_\omega \; \forall s \in S \;\; b >_\omega s$
- $\forall \alpha, \beta \in \mathcal{HR}_\omega$ s.t. $(\beta >_\omega \alpha)$ we have $(\forall s \in S \;\; \beta \geq_\omega s)$ or $(\exists s \in S \;\; s >_\omega \alpha)$

Then S has a least-upper bound in $\mathcal{HR}_\omega$.

This last property is proved by an iterative trichotomy process applied on an initial interval $[s_0, b_0]$ of $\mathcal{HR}_\omega$ such that $s_0 \in S$ and b_0 is an upper bound of S in $\mathcal{HR}_\omega$.

Critique et perspective de constructivité

- Le constructivisme voit l'existence comme une construction alors qu'en nonstandard, les infiniment grands n'existent que par convention.
- Ce n'est qu'une fois le jeu du nonstandard accepté qu'$\mathcal{HR}_\omega$ vérifie les critères de constructivité.
- On se propose de remplacer les entiers nonstandard axiomatiques par ceux introduits par Schmieden et Laugwitz.

La théorie de Laugwitz

L'idée générale de Laugwitz est l'extention d'une théorie T en une nouvelle notée $T < \Omega >$.
On rajoute une entité Ω à l'ensemble des entiers de $\mathbb{N}$. Elle est caractérisée par :

Règle Basique (RB)

Soit $S(n)$ une formule sur $\mathbb{N}$. Si, à partir d'un certain rang, $S(n)$ est vraie pour tous les entiers, alors $S(\Omega)$ est vraie.

Ω, infiniment grand

Propriété

Ω est infiniment grand.

◁

- "Ω infiniment grand" signifie $\Omega > p$ pour chaque p entier
- à partir de $n_0 = p + 1$, on a bien $n > p$ pour tout $n \in \mathbb{N}$
- de la règle basique, on conclut que pour tout p, $\Omega > p$

▷

La Règle Basique précise le principe leibnizien selon lequel les *"lois du fini réussissent dans l'infini"* à l'égard d'Ω.

Relation d'équivalence

Définition usuelle

Soit $S(n)$ et $T(n)$ deux suites d'entiers, on dit que ces deux suites sont égales si elles le sont termes à termes.

Définition nuancée

Soit $S(n)$ et $T(n)$ deux suites d'entiers, on dit que ces deux suites sont égales si, à partir d'un certain rang elles sont égales.

Notation

$S(\Omega) =_0 T(\Omega)$ s'il existe n_0 tel que pour tout $n > n_0$, $S(n) = T(n)$.

Les $\Omega-$Entiers

Cette relation d'équivalence permet la définition suivante :

Définition des Ω-Entiers

Les Ω-Entiers sont les classes d'équivalence de suites pour la relation d'égalité $=_0$ définie ci-dessus, cet ensemble est noté $\mathbb{Z}_\Omega$.

- La suite constante $n \longrightarrow a \in \mathbb{Z}$ définit l' Ω-Entiers (standard) a.

Propriété

Il existe des éléments nonstandard dans $\mathbb{Z}_\Omega$.

- La suite $n \longrightarrow n$ définit l' Ω-Entiers (nonstandard) Ω.

Caractéristique de la théorie

Pour chaque couple d'entiers (a, b) de $\mathbb{N}^2$, la propriété suivante est vraie :

$$a > b \quad \vee \quad a = b \quad \vee \quad a < b$$

Pour chaque suite (a_n, b_n), la propriété suivante est vraie à chaque rang :

$$a_n > b_n \quad \vee \quad a_n = b_n \quad \vee \quad a_n < b_n$$

Ainsi, d'après la Règle Basique, pour chaque couple d'Ω-Entiers (a_Ω, b_Ω), la propriété suivante est vraie :

$$a_\Omega > b_\Omega \quad \vee \quad a_\Omega = b_\Omega \quad \vee \quad a_\Omega < b_\Omega \quad (\star)$$

Mais on peut très bien avoir pour $i \neq j \neq k$:

$$a_i > b_i \quad \vee \quad a_j = b_j \quad \vee \quad a_k < b_k$$

et ainsi ne pas savoir de quelle manière passer de $(\star)$ à $a_\Omega > b_\Omega$ ou bien $a_\Omega = b_\Omega$ ou bien $a_\Omega < b_\Omega$.

Les Ω-Entiers comme base de construction de $\mathcal{HR}_\Omega$

A partir de ces entiers nonstandard de $\mathbb{Z}_\Omega$, il est possible de reconstruire la droite d'Harthong-Reeb.
On va utiliser l'échelle fournie par Laugwitz : Ω.

$$\mathcal{HR}_\Omega = \{X \in \mathbb{Z}_\Omega,\ \exists n \in \mathbb{N},\ |X| \leq n\Omega\}$$

Soient $x = (x_m)$ et $y = (y_m)$ des éléments quelconques de $\mathcal{HR}_\Omega$.

$$\begin{array}{rcc} x =_\Omega y & \Leftrightarrow & \text{pour chaque n de } \mathbb{N},\ n|x-y| \leq_0 \Omega \\ & \Leftrightarrow & \text{pour chaque n de } \mathbb{N},\ \exists M_n \in \mathbb{N} \text{ tel que } \forall m > M_n, \\ & & n|x_m - y_m| \leq m \\ x >_\Omega y & \Leftrightarrow & \text{il existe n de } \mathbb{N},\ n(x-y) \geq_0 \Omega \\ & \Leftrightarrow & \text{il existe n de } \mathbb{N}, \exists M_n \in \mathbb{N} \text{ tel que } \forall m > M_n, \\ & & n(x_m - y_m) \geq m \end{array}$$

$\mathcal{HR}_\Omega$ vérifie-t-elle les axiomes de Brigdes ?

- Les premiers axiomes de Bridges se vérifient sans difficulté.
- L'axiome de la borne supérieure est un peu plus délicat à vérifier.
- Une meilleure contructivité de ce modèle est indéniable car on sait travailler avec Ω alors qu'ω n'était qu'axiomatique.
- Comment arriver à de la programmation ?

Conclusion

- This work is a first look into a formal and constructive approach to discrete geometry.
- The nonstandard origin of discrete analytical geometry is recalled.
- We show that the Harthong-Reeb line is a model of Bridges-Heyting ordered field. $\mathcal{HR}_\omega$ is not constructive but we can work in a constructive manner with it.
- L'approche "Harthong-Reeb" peut être adaptée à d'autres théories.

Future work

- This work is pursued into two main directions:
  - Work on $\mathcal{HR}_\omega$ like models that are better suited for a constructive approach.
  - Propose other arithmetization of integration schematas. We are right now working on digital circles.
- $\mathcal{HR}_\omega$ is a discrete model of the continuum. We hope it could be used as a framework for discrete differential and discrete parametric geometry.